Maximum Twist Width of Delta-Matroids

Logan Keck

July 6, 2026

Table of Contents

1. Review
2. Maximum Twist Width
3. Monotonicity and Maximum Twist Width
4. Algorithm

Table of Contents

1. Review
2. Maximum Twist Width
3. Monotonicity and Maximum Twist Width
4. Algorithm

Definitions

Definition

A ribbon graph is a graph where vertices are disks and edges are ribbons connecting those vertices, possibly with twists.

Definitions

Definition

A ribbon graph is a graph where vertices are disks and edges are ribbons connecting those vertices, possibly with twists.

Definition

Given ribbon graph G and $A \subseteq E(G)$, the partial dual of G with respect to A, denoted G^A, is the ribbon graph obtained by gluing disks to each boundary component of the spanning ribbon subgraph $(V(G), A)$, where these are the vertices of G^A, removing the vertices of G, and keeping edges unchanged.

Definitions

Definition

A ribbon graph is a graph where vertices are disks and edges are ribbons connecting those vertices, possibly with twists.

Definition

Given ribbon graph G and $A \subseteq E(G)$, the partial dual of G with respect to A, denoted G^A, is the ribbon graph obtained by gluing disks to each boundary component of the spanning ribbon subgraph $(V(G), A)$, where these are the vertices of G^A, removing the vertices of G, and keeping edges unchanged.

Partial duals can also be thought of using arrows or by a table.

Definitions

Definition

A *delta-matroid* is a proper set system $D = (E, \mathcal{F})$ which satisfies the symmertic exchange axiom:
For any $X, Y \in \mathcal{F}$ and any $u \in X \Delta Y$, there exists a $v \in X \Delta Y$ s.t. $X \Delta \{u, v\} \in \mathcal{F}$.

Definitions

Definition

A *delta-matroid* is a proper set system $D = (E, \mathcal{F})$ which satisfies the symmertic exchange axiom:
For any $X, Y \in \mathcal{F}$ and any $u \in X \Delta Y$, there exists a $v \in X \Delta Y$ s.t. $X \Delta \{u, v\} \in \mathcal{F}$.

Definition

For set system $D = (E, \mathcal{F})$ and $A \subseteq E$, the *twist* of D with respect to A, denoted $D \Delta A$, is the set system $(E, \mathcal{F} \Delta A)$ where

$$F \Delta A := \{X \Delta A : X \in \mathcal{F}\}.$$

Definitions

Definition

Given a ribbon graph $G = (V(G), E(G))$, define

$$\mathcal{F}(G) := \{F \subseteq E(G) : F \text{ is the edge set of a spanning quasi-tree of } G\}.$$

Then define the *delta-matroid* of G as $D(G) := (E(G), \mathcal{F}(G))$

Recall that partial duals and twists are closely related, as

$$D(G^A) = D(G)\Delta A.$$

Table of Contents

1. Review
2. Maximum Twist Width
3. Monotonicity and Maximum Twist Width
4. Algorithm

And More Definitions

Definition

Given a set system $D = (E, \mathcal{F})$, denote by $\mathcal{F}_{max}(D)$ and $\mathcal{F}_{min}(D)$ the collections of feasible sets of maximum and minimum cardinality, respectively. Denote by $r_{max}(D)$ and $r_{min}(D)$ the cardinalities of a maximum and minimum feasible set, respectively.

Definition

Given a set system $D = (E, \mathcal{F})$, define the *width* of D as

$$\omega(D) := r_{max}(D) - r_{min}(D).$$

And More Definitions

Definition

Given a set system $D = (E, \mathcal{F})$, denote by $\mathcal{F}_{max}(D)$ and $\mathcal{F}_{min}(D)$ the collections of feasible sets of maximum and minimum cardinality, respectively. Denote by $r_{max}(D)$ and $r_{min}(D)$ the cardinalities of a maximum and minimum feasible set, respectively.

Definition

Given a set system $D = (E, \mathcal{F})$, define the *width* of D as

$$\omega(D) := r_{max}(D) - r_{min}(D).$$

Definition

Given a set system $D = (E, \mathcal{F})$, define the *maximum twist width* of D by

$$\partial\omega_M(D) = \max\{\omega(D \Delta A) \mid A \subseteq E\}.$$

And More Definitions

Definition

Given a set system $D = (E, \mathcal{F})$, denote by $\mathcal{F}_{max}(D)$ and $\mathcal{F}_{min}(D)$ the collections of feasible sets of maximum and minimum cardinality, respectively. Denote by $r_{max}(D)$ and $r_{min}(D)$ the cardinalities of a maximum and minimum feasible set, respectively.

Definition

Given a set system $D = (E, \mathcal{F})$, define the *width* of D as

$$\omega(D) := r_{max}(D) - r_{min}(D).$$

Definition

Given a set system $D = (E, \mathcal{F})$, define the *maximum twist width* of D by

$$\partial\omega_M(D) = \max\{\omega(D \Delta A) \mid A \subseteq E\}.$$

Maximum Twist Width

Our goal is to prove the following theorem:

Theorem

For any delta-matroid $D = (E, \mathcal{F})$, there is a sequence of elements $e_1, e_2, \ldots, e_k$ such that

$$\omega(D\Delta\{e_1, e_2, \ldots, e_k\}) = \partial\omega_M(D)$$

and such that the sequence

$$\omega(D), \omega(D\Delta\{e_1\}), \ldots, \omega(D\Delta\{e_1, e_2, \ldots, e_k\})$$

rises monotonically (never decreasing) to $\partial\omega_M(D)$.

Connection to Euler Genus

Recall that the Euler genus of a ribbon graph G, $\gamma(G)$, satisfies $\gamma(G) = \omega(D(G))$. Define $\partial\gamma_M(G) := \max\{\gamma(G^A) \mid A \subseteq E(G)\}$. Then as a corollary, we will get the following:

Connection to Euler Genus

Recall that the Euler genus of a ribbon graph G, $\gamma(G)$, satisfies $\gamma(G) = \omega(D(G))$. Define $\partial\gamma_M(G) := \max\{\gamma(G^A) \mid A \subseteq E(G)\}$. Then as a corollary, we will get the following:

Corollary

For any ribbon graph G, there is a sequence of edges $e_1, e_2, \ldots, e_k$ such that

$$\gamma(G^{\{e_1,e_2,\ldots,e_k\}}) = \partial\gamma_M(G)$$

and such that the sequence

$$\gamma(G), \gamma\left(G^{\{e_1\}}\right), \ldots, \gamma\left(G^{\{e_1,e_2,\ldots,e_k\}}\right)$$

rises monotonically (never decreasing) to $\partial\gamma_M(G)$.

A Property of the Maximum Twist Width

Theorem

For any proper set system $D = (E, \mathcal{F})$, there exists a feasible set $F \in \mathcal{F}$ such that

$$\omega(D \Delta F) = \partial\omega_M(D).$$

A Property of the Maximum Twist Width

Theorem

For any proper set system $D = (E, \mathcal{F})$, there exists a feasible set $F \in \mathcal{F}$ such that

$$\omega(D \Delta F) = \partial\omega_M(D).$$

Proof. It is sufficient to show that for all $A \subseteq E$, there is some $F \in \mathcal{F}$ such that $\omega(D\Delta F) \geq \omega(D\Delta A)$. If $A \in \mathcal{F}$, taking $F = A$ completes the proof, so assume that $A \notin \mathcal{F}$.

Choose an $F \in \mathcal{F}$ that minimizes $|A\Delta F|$. Note that $A\Delta F \neq \emptyset$. For each $f \in F\Delta A$,

$$\begin{aligned} r_{\min}(D\Delta\{A\Delta f\}) &= r_{\min}((D\Delta A)\Delta f) = |\{F\Delta A\}\Delta f| \\ &= |F\Delta A| - 1 = r_{\min}(D\Delta A) - 1 \end{aligned}$$

and

$$r_{\max}(D\Delta\{A\Delta f\}) \geq r_{\max}(D\Delta A) - 1.$$

A Property of the Maximum Twist Width cont'd

Therefore, $\omega(D\Delta\{A\Delta f\}) \geq \omega(D\Delta A)$. Repeating this process for all elements in $F\Delta A$ yields

$$\omega(D\Delta F) = \omega(D\Delta A\Delta\{A\Delta F\}) \geq \omega(D\Delta A).$$

And so we are done.

Therefore, $\omega(D\Delta\{A\Delta f\}) \geq \omega(D\Delta A)$. Repeating this process for all elements in $F\Delta A$ yields

$$\omega(D\Delta F) = \omega(D\Delta A\Delta\{A\Delta F\}) \geq \omega(D\Delta A).$$

And so we are done. As a corollary for ribbon graphs, we get

Corollary

For any ribbon graph G, there exists a spanning quasi-tree A of G such that

$$\gamma(G^A) = \partial\gamma_M(G).$$

Partial-Duality Deficiency

Given ribbon graph G and $A \subseteq E(G)$, the following formula holds:

$$\gamma(G^A) = 2 + e(G) - f(A) - f(A^c).$$

When A is the edge set of a spanning quasi-tree, we have $f(A) = 1$ and so

$$\gamma(G^A) = 1 + e(G) - f(A^c).$$

Partial-Duality Deficiency

Given ribbon graph G and $A \subseteq E(G)$, the following formula holds:

$$\gamma(G^A) = 2 + e(G) - f(A) - f(A^c).$$

When A is the edge set of a spanning quasi-tree, we have $f(A) = 1$ and so

$$\gamma(G^A) = 1 + e(G) - f(A^c).$$

So, $\gamma(G^A)$ attains its maximum if and only if $f(A^c)$ attains its minimum.

Partial-Duality Deficiency

Given ribbon graph G and $A \subseteq E(G)$, the following formula holds:

$$\gamma(G^A) = 2 + e(G) - f(A) - f(A^c).$$

When A is the edge set of a spanning quasi-tree, we have $f(A) = 1$ and so

$$\gamma(G^A) = 1 + e(G) - f(A^c).$$

So, $\gamma(G^A)$ attains its maximum if and only if $f(A^c)$ attains its minimum.

Proposition

Let G be a connected ribbon graph and let $\mathcal{T}$ denote the set of spanning quasi-trees of G. Then

$$\partial\gamma_M(G) = 1 + e(G) - \min_{A \in \mathcal{T}} f(A^c).$$

Partial-Duality Deficiency cont'd

The invariant $\min_{A \in \mathcal{T}} f(A^c)$ is called the *partial-duality deficiency* of G, denoted $\partial\xi(G)$.

Partial-Duality Deficiency cont'd

The invariant $\min_{A \in \mathcal{T}} f(A^c)$ is called the *partial-duality deficiency* of G, denoted $\partial\xi(G)$. Given a set system $D = (E, \mathcal{F})$, we get that

$$\partial\omega_M(D) = \max\{|F_1 \Delta F_2| \mid F_1, F_2 \in \mathcal{F}\}.$$

Combining this with the previous proposition, we get

Proposition

Let G be a connected ribbon graph and $D(G) = (E(G), \mathcal{F}(G))$ be its delta-matroid. Then

$$\partial\xi(G) = e(G) + 1 - \max\{|F_1 \Delta F_2| \mid F_1, F_2 \in \mathcal{F}(G)\}.$$

Table of Contents

1 Review

2 Maximum Twist Width

3 Monotonicity and Maximum Twist Width

4 Algorithm

Main Theorem

It's time to prove our main theorem. But first, a helpful lemma:

Lemma

For any delta-matroid $D = (E, \mathcal{F})$ and any $F_0 \in \mathcal{F}$, there exist $F_1 \in \mathcal{F}_{\min}(D)$ and $F_2 \in \mathcal{F}_{\max}(D)$ such that $F_1 \subseteq F_0 \subseteq F_2$.

Main Theorem

It's time to prove our main theorem. But first, a helpful lemma:

Lemma

For any delta-matroid $D = (E, \mathcal{F})$ and any $F_0 \in \mathcal{F}$, there exist $F_1 \in \mathcal{F}_{\min}(D)$ and $F_2 \in \mathcal{F}_{\max}(D)$ such that $F_1 \subseteq F_0 \subseteq F_2$.

Here is what we will prove now (slightly stronger than previously stated):

Theorem

For any delta-matroid $D = (E, \mathcal{F})$, there exists a sequence of elements $e_1, e_2, \ldots, e_k$ in E such that $\{e_1, e_2, \ldots, e_k\} \in \mathcal{F}$, $\omega(D\Delta\{e_1, e_2, \ldots, e_k\}) = \partial\omega_M(D)$, and the sequence of widths

$$\omega(D), \omega(D\Delta e_1), \ldots, \omega(D\Delta\{e_1, e_2, \ldots, e_k\})$$

increases monotonically to $\partial\omega_M(D)$.

Proof of Main Theorem

Let $\hat{\mathcal{F}}$ be the set of feasible sets $A \in \mathcal{F}$ such that $\omega(D\Delta A) = \partial\omega_M(D)$. By our previous theorem, $\hat{\mathcal{F}}$ is nonempty.

Proof of Main Theorem

Let $\hat{\mathcal{F}}$ be the set of feasible sets $A \in \mathcal{F}$ such that $\omega(D\Delta A) = \partial\omega_M(D)$. By our previous theorem, $\hat{\mathcal{F}}$ is nonempty. Let F be a feasible set of minimum cardinality in $\hat{\mathcal{F}}$. By this choice, F satisfies the conclusions of the theorem, except possibly for the fact that taking a sequence of widths increases monotonically. We will be done if we can show that a sequence rises monotonically for F.

Proof of Main Theorem

Let $\hat{\mathcal{F}}$ be the set of feasible sets $A \in \mathcal{F}$ such that $\omega(D \Delta A) = \partial\omega_M(D)$. By our previous theorem, $\hat{\mathcal{F}}$ is nonempty. Let F be a feasible set of minimum cardinality in $\hat{\mathcal{F}}$. By this choice, F satisfies the conclusions of the theorem, except possibly for the fact that taking a sequence of widths increases monotonically. We will be done if we can show that a sequence rises monotonically for F.

If $|F| = 0$, we are done. Assume that the conclusion holds when $|F| = k$. We will try to show that it holds then for $|F| = k + 1$, proceeding by induction. By our lemma, there exist $F_1 \in \mathcal{F}_{\text{min}}(D)$ and $F_2 \in \mathcal{F}_{\text{max}}(D)$ such that

$$F_1 \subseteq F \subseteq F_2.$$

Proof of Main Theorem: Case 1

Case 1. If $F_1 \neq \emptyset$, let $f \in F_1 \subseteq F$. Then

$$r_{\min}(D\Delta f) = |F_1 \Delta f| = r_{\min}(D) - 1$$

and

$$r_{\max}(D\Delta f) \geq r_{\max}(D) - 1.$$

So, $\omega(D\Delta f) \geq \omega(D)$.

Proof of Main Theorem: Case 1

Case 1. If $F_1 \neq \emptyset$, let $f \in F_1 \subseteq F$. Then

$$r_{\min}(D \Delta f) = |F_1 \Delta f| = r_{\min}(D) - 1$$

and

$$r_{\max}(D \Delta f) \geq r_{\max}(D) - 1.$$

So, $\omega(D \Delta f) \geq \omega(D)$.
Now, let $D' = D \Delta f$ and $\hat{\mathcal{F}}'$ be the set of feasible sets of D' such that $\omega(D' \Delta A) = \partial\omega_M(D')$ for every $A \in \hat{\mathcal{F}}'$. By definition, $\partial\omega_M(D') = \partial\omega_M(D)$. I claim that $F \Delta f$ is a minimum cardinality feasible set in $\hat{\mathcal{F}}'$.

Proof of Main Theorem: Case 1

Case 1. If $F_1 \neq \emptyset$, let $f \in F_1 \subseteq F$. Then

$$r_{\min}(D\Delta f) = |F_1 \Delta f| = r_{\min}(D) - 1$$

and

$$r_{\max}(D\Delta f) \geq r_{\max}(D) - 1.$$

So, $\omega(D\Delta f) \geq \omega(D)$.
Now, let $D' = D\Delta f$ and $\hat{\mathcal{F}}'$ be the set of feasible sets of D' such that $\omega(D'\Delta A) = \partial\omega_M(D')$ for every $A \in \hat{\mathcal{F}}'$. By definition, $\partial\omega_M(D') = \partial\omega_M(D)$. I claim that $F\Delta f$ is a minimum cardinality feasible set in $\hat{\mathcal{F}}'$. Since $\omega(D\Delta F) = \partial\omega_M(D)$, there exists a feasible set X of D such that $|F\Delta X| = \partial\omega_M(D)$.

Proof of Main Theorem: Case 1 cont'd

Now,

$$|(F\Delta f)\Delta(X\Delta f)| = \partial\omega_M(D) = \partial\omega_M(D'),$$

where $X\Delta f$ is a feasible set of D'. So, $F\Delta f \in \hat{\mathcal{F}}'$.

Proof of Main Theorem: Case 1 cont'd

Now,

$$|(F\Delta f)\Delta(X\Delta f)| = \partial\omega_M(D) = \partial\omega_M(D'),$$

where $X\Delta f$ is a feasible set of D'. So, $F\Delta f \in \hat{\mathcal{F}}'$. We still need to show that $F\Delta f$ has minimum cardinality in $\hat{\mathcal{F}}'$. Assume for the sake of contradiction that there is a $T \in \hat{\mathcal{F}}'$ such that $|T| < |F\Delta f|$. Then $T\Delta f \in \hat{\mathcal{F}}$ and $|T\Delta f| < |F|$, which contradicts the fact that F has minimum cardinality in $\hat{\mathcal{F}}$.
Since $|F\Delta f| = k$, the conclusion follows by the inductive hypothesis.

Proof of Main Theorem: Case 2

Case 2. Assume $F_1 = \emptyset$.
Case 2.1. For some $F_3 \in \mathcal{F}_{\max}(D), F \not\subseteq F_3$. So, there exists an $f \in F$ and $f \notin F_3$. Then

$$r_{\max}(D \Delta f) = |F_3 \Delta f| = r_{\max}(D) + 1$$

and

$$r_{\min}(D \Delta f) \leq r_{\min}(D) + 1.$$

Proof of Main Theorem: Case 2

Case 2. Assume $F_1 = \emptyset$.
Case 2.1. For some $F_3 \in \mathcal{F}_{\max}(D), F \not\subseteq F_3$. So, there exists an $f \in F$ and $f \notin F_3$. Then

$$r_{\max}(D \Delta f) = |F_3 \Delta f| = r_{\max}(D) + 1$$

and

$$r_{\min}(D \Delta f) \leq r_{\min}(D) + 1.$$

So, $\omega(D \Delta f) \geq \omega(D)$. Let $D' = D \Delta f$ and $\hat{\mathcal{F}}'$ be the set of feasible sets of D' such that $\omega(D' \Delta F) = \partial\omega_M(D')$ holds for every $F \in \hat{\mathcal{F}}'$. Then $F \Delta f$ is a feasible set of minimum cardinality in $\hat{\mathcal{F}}'$, with the proof being similar to that in Case 1. Since $|F \Delta f| = k$, the conclusion holds by the inductive hypothesis.

Proof of Main Theorem: Case 2 cont'd

Case 2.2. For any $A \in \mathcal{F}_{\max}(D)$, $F \subseteq A$. Since $\omega(D \Delta F) = \partial\omega_M(D)$, there exists a feasible set X with $|F \Delta X| = \partial\omega_M(D)$. By our lemma, there is some $F_3 \in \mathcal{F}_{\max}(D)$ such that $X \subseteq F_3$. Since $\emptyset = F_1 \in \mathcal{F}$, $\omega(D) = r_{\max} - 0 = |F_3|$, and since $F \neq \emptyset$, we know $\emptyset \notin \hat{\mathcal{F}}$.

Proof of Main Theorem: Case 2 cont'd

Case 2.2. For any $A \in \mathcal{F}_{\max}(D)$, $F \subseteq A$. Since $\omega(D\Delta F) = \partial\omega_M(D)$, there exists a feasible set X with $|F\Delta X| = \partial\omega_M(D)$. By our lemma, there is some $F_3 \in \mathcal{F}_{\max}(D)$ such that $X \subseteq F_3$. Since $\emptyset = F_1 \in \mathcal{F}$, $\omega(D) = r_{\max} - 0 = |F_3|$, and since $F \neq \emptyset$, we know $\emptyset \notin \hat{\mathcal{F}}$. Then

$$|F\Delta X| = \partial\omega_M(D) > \omega(D) = |F_3|.$$

Since $F_3 \in \mathcal{F}_{\max}(D)$, $F \subseteq F_3$. So, we have $F \cup X \subseteq F_3$, and also

$$|F_3| \geq |F \cup X| \geq |F\Delta X| > |F_3|,$$

which is impossible. The proof is complete.

A Remark

The fact that we use a delta-matroid does matter; the main theorem doesn't hold generally for set systems. For example, take $D = ([5], \{\emptyset, \{1,2\}, \{3,4,5\}, \{1,2,3,4\}\})$. It is clear that $\omega(D) = 4$, and it's not hard to check that only $D\Delta\{1,2\}$ and $D\Delta\{3,4,5\}$ have the maximum twist width of D, that being 5. So, no sequence rises monotonically to $\partial\omega_M(D)$.

Table of Contents

1 Review

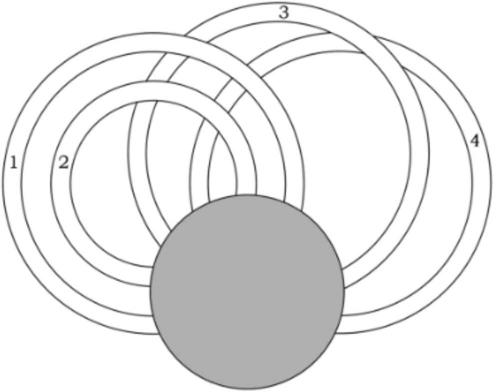
2 Maximum Twist Width

3 Monotonicity and Maximum Twist Width

4 Algorithm

Algorithm Example

The proof of the main theorem essentially provides an algorithm for constructing a sequence satisfying the monotonicity property in that theorem. We will apply this algorithm to a ribbon graph.



Algorithm Example cont'd

The delta-matroid of G is $D(G) = ([4], \mathcal{F})$, where $\mathcal{F} = \{\emptyset, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}$. It is clear that $\hat{\mathcal{F}} = \{\{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}\}$. We can start the algorithm with $F = \{1,3\}$ as a minimum cardinality element of $\hat{\mathcal{F}}$.

Algorithm Example cont'd

The delta-matroid of G is $D(G) = ([4], \mathcal{F})$, where
$\mathcal{F} = \{\emptyset, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}$. It is clear that
$\hat{\mathcal{F}} = \{\{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}\}$. We can start the algorithm with
$F = \{1,3\}$ as a minimum cardinality element of $\hat{\mathcal{F}}$.
First Iteration: Since $\emptyset \in \mathcal{F}$, select $X = \{2,3\} \in \mathcal{F}_{\text{max}}$.
Choose $1 \in F \setminus X$.
Set $S = [1]$, $F' = F \setminus \{1\} = \{3\}$,
$\mathcal{F}' = \{\{1\}, \{3\}, \{4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}\}$.

Algorithm Example cont'd

The delta-matroid of G is $D(G) = ([4], \mathcal{F})$, where
$\mathcal{F} = \{\emptyset, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}$. It is clear that
$\hat{\mathcal{F}} = \{\{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}\}$. We can start the algorithm with
$F = \{1,3\}$ as a minimum cardinality element of $\hat{\mathcal{F}}$.
First Iteration: Since $\emptyset \in \mathcal{F}$, select $X = \{2,3\} \in \mathcal{F}_{\text{max}}$.
Choose $1 \in F \setminus X$.
Set $S = [1]$, $F' = F \setminus \{1\} = \{3\}$,
$\mathcal{F}' = \{\{1\}, \{3\}, \{4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}\}$.
Second Iteration: Since $\emptyset \notin \mathcal{F}'$, select $X = \{3\} \in \mathcal{F}'_{\text{min}}$.
Choose $3 \in X \subseteq F'$.
Set $S = [1,3]$, $F'' = F' \setminus \{3\} = \emptyset$,
$\mathcal{F}'' = \{\emptyset, \{1,2\}, \{1,3\}, \{1,4\}, \{3,4\}, \{1,2,3,4\}\}$.

Algorithm Example cont'd

The delta-matroid of G is $D(G) = ([4], \mathcal{F})$, where
$\mathcal{F} = \{\emptyset, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}$. It is clear that
$\hat{\mathcal{F}} = \{\{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}\}$. We can start the algorithm with
$F = \{1,3\}$ as a minimum cardinality element of $\hat{\mathcal{F}}$.
First Iteration: Since $\emptyset \in \mathcal{F}$, select $X = \{2,3\} \in \mathcal{F}_{\max}$.
Choose $1 \in F \setminus X$.
Set $S = [1]$, $F' = F \setminus \{1\} = \{3\}$,
$\mathcal{F}' = \{\{1\}, \{3\}, \{4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}\}$.
Second Iteration: Since $\emptyset \notin \mathcal{F}'$, select $X = \{3\} \in \mathcal{F}'_{\min}$.
Choose $3 \in X \subseteq F'$.
Set $S = [1,3]$, $F'' = F' \setminus \{3\} = \emptyset$,
$\mathcal{F}'' = \{\emptyset, \{1,2\}, \{1,3\}, \{1,4\}, \{3,4\}, \{1,2,3,4\}\}$.
Termination: Since $F'' = \emptyset$, return $S = [1,3]$.

Refrences

X. Jin, Z. Li, Q. Yan, G. Zhang, On the maximum twist width of delta-matroids, Preprint arXiv:2602.01946v1 [math.CO]